Resonance of the thermal boundary layer adjacent to an isothermally heated vertical surface

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The instability characteristics and resonance of a natural convection boundary layer adjacent to an isothermally heated vertical surface are investigated using direct stability analyses. The detailed streamwise evolution of the boundary-layer frequencies is visualized via the power spectra of the temperature time series in the thermal boundary layer. It is found that the entire thermal boundary layer may be divided into three distinct regions according to the frequency profile, which include an upstream low-frequency region, a transitional region (with both low- and high-frequency bands) and a downstream high-frequency region. The high-frequency band in the downstream region determines the resonance characteristics of the thermal boundary layer, which can be triggered by a single-mode perturbation at frequencies within the high-frequency band. The single-mode perturbation experiments further reveal that the maximum resonance of the thermal boundary layer is triggered by a perturbation at the characteristic frequency of the boundary layer. For the boundary-layer flow at $Ra = 3.6 \times 10^{10}$ and $Pr = 7$, a net heat transfer enhancement of up to 44% is achieved by triggering resonance of the boundary layer. This significant enhancement of heat transfer is due to the resonance-induced advancement of the laminar–turbulent transition, which is found to be dependent on the perturbation frequency and amplitude. Evidence from different perspectives revealing the same position of the transition are provided and discussed. The outcomes of this investigation demonstrate the prospect of a resonance-based approach for enhancing heat transfer.

Key words: buoyant boundary layers, convection, instability control

1. Introduction

Natural convection is a common flow phenomenon that arises from thermal surfaces such as the surface of the Earth, computer heat-sinks and the skin of the human body. In the past decades extensive investigations into this flow behaviour have been conducted for the purpose of understanding the mechanisms and exploring the engineering applications of the flow. Among those investigations, many were concerned with the properties of the natural convection boundary layer adjacent to a heated wall, which determines heat dissipation rate. A particular topic of continued

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research interest is the stability characteristics of the thermal boundary layer, because of its importance for understanding the laminar–turbulent transition.

The natural convection flow adjacent to a vertical heated surface has received much attention. Research into the instability of the thermal boundary-layer flow over a flat vertical plate began in the 1950s. One of the pioneering experimental studies was performed by Eckert & Soehngen (1951), in which a Mach–Zehnder interferometer was used to observe the laminar boundary-layer flow in air subjected to natural disturbances. The experiments revealed that the thermal boundary layer amplified a certain range of the initial small disturbances and finally became turbulent. The disturbances were initially amplified in a two-dimensional sinusoidal form and later burst into more complex waves. Holman, Gartrell & Soehngen (1960) later performed experiments with artificially introduced perturbations and found that the thermal boundary layer amplified a particular disturbance frequency the most. Polymeropoulos & Gebhart (1967) experimentally studied the evolution of artificial disturbances, generated by an oscillating ribbon, in the natural convection boundary layer adjacent to a uniformly heated vertical plate. Four evolution modes, i.e. amplified, damped, damped–amplified (switch from decaying to amplifying) and amplified–damped (switch from amplifying to decaying), were identified, which were considered to be dependent on the ribbon oscillation frequency and the local stability characteristics of the boundary layer. Similar experiments by Knowles & Gebhart (1968) and Dring & Gebhart (1968) also found that the thermal boundary-layer flow only favoured a narrow band of disturbance frequencies for amplification. The above-mentioned experiments imply that natural convection boundary-layer flows sharply filter out disturbances, keeping and amplifying only disturbances over a certain frequency band. A review of this frequency-filtering effect of the thermal boundary layer is available in Gebhart (1973).

In theoretical analyses, a fundamental assumption of the linear stability theory is that the laminar base flow is perturbed by infinitesimal velocity and temperature disturbances (see e.g. Plapp 1957; Szewczyk 1962; Nachtsheim 1963). The disturbances are usually assumed to be two-dimensional and mathematically decomposed into similar periodic components. The evolution process of the disturbances, being either damped or amplified, depends on whether the disturbances contribute to or absorb energy from the base flow. On the basis of the linear stability theory, Dring & Gebhart (1968) investigated disturbance amplification of the natural convection boundary layer adjacent to a vertical flat surface with a uniform heat flux by numerically integrating the coupled Orr–Sommerfeld disturbance equations. Their investigation revealed that the instability of the thermal boundary layer was dominated by high-frequency disturbances when the boundary layer was exposed to both low- and high-frequency disturbances. Further studies by Gebhart & Mahajan (1975) concluded that the characteristic frequency of a vertical natural convection boundary-layer flow only depended on the heating condition and the Prandtl number \( Pr \) (\( Pr = \nu/\kappa \), where \( \nu \) and \( \kappa \) are the kinematic viscosity and thermal diffusivity of the fluid, respectively). They also estimated the characteristic frequencies of the thermal boundary-layer flows under both isothermal and isoflux conditions. Their semi-analytical estimation for the isothermal boundary-layer flow is expressed as \( f' = B (g \beta \Delta T)^{2/3} (2\pi)^{-1} \), where \( f' \) is the dimensional characteristic frequency, \( B = 0.315 Pr^{-0.065} \) was derived by correlating experimental data of the isothermal boundary-layer flow, \( \beta \) is the thermal expansion coefficient of the fluid, \( g \) is the gravitational acceleration, and \( \Delta T \) is the constant heating temperature difference.
In addition to the experimental and theoretical studies, direct stability analysis is another powerful approach for studying wave properties and instabilities of natural convection boundary layers. Here, the perturbed governing equations or the unperturbed governing equations with perturbed boundary conditions are solved numerically and the perturbation characteristics analysed for growth or decay. This approach has been widely applied in recent years, for example by Armfield & Patterson (1992), Janssen & Henkes (1995), Armfield & Janssen (1996), Janssen & Armfield (1996), Brooker, Patterson & Graham (2000), Lei & Patterson (2003), Aberra et al. (2006), Paul & Rees (2008), Paul, Rees & Wilson (2008, 2010), Williamson, Armfield & Kirkpatrick (2012) and Aberra et al. (2012). Some of these studies were concerned with the instability properties of the natural convection boundary layer in a differentially heated cavity. Reviews of these studies are available in Janssen & Henkes (1995) and Williamson et al. (2012).

The direct stability analysis approach has also been adopted to investigate the instability characteristics of the natural convection boundary layer on a vertical flat surface. For instance, Paul & Rees (2008) and Paul et al. (2008, 2010) studied the linear and nonlinear instabilities of free convection boundary-layer flows with Prandtl numbers of 0.7 and 1, respectively. Their investigations of the linear instability revealed that the most dangerous (unstable) frequency in the streamwise direction increases with the distance from the leading edge. Here the most dangerous frequency refers to the perturbation frequency at which the local maximum response of the boundary layer can be triggered. Paul et al. (2010) later extended their investigation into the nonlinear regime by seeding disturbances of finite amplitude. It was found that the nonlinear response of the boundary layer largely depends on the frequency and amplitude of the disturbance. The nonlinearity was found to be present in the form of cells (formed by isothermal lines) splitting and merging. Aberra et al. (2006, 2012) examined the critical Rayleigh number of the convective instability in buoyancy-driven flows with an isoflux boundary condition. The latter paper also examined the instability of the boundary-layer flow with Neumann and Dirichlet disturbance boundary conditions, respectively. It was found that the different disturbance boundary conditions produced similar numerical stability results. The critical Rayleigh number of the convective instability was numerically determined to be $Ra \approx 9.2 \times 10^6$ for $Pr = 6.7$. The convective instability in a time-dependent one-dimensional buoyant flow was also studied by Brooker et al. (2000). It was demonstrated that the instability could be responsible for the breakdown of the one-dimensional flow before the arrival of the leading-edge effect.

 Whilst the instability properties of natural convection boundary layers have been investigated extensively, the resonance of natural convection boundary layers has attracted much less research interest. Resonance of natural convection flows in a cavity was initially reported by Lage & Bejan (1993), followed by a few other investigations (e.g. Kwak & Hyun 1996; Kwak, Kuwahara & Hyun 1998). Among these investigations, some were concerned with external periodical-heating-induced resonance of natural convection (e.g. Kim, Kim & Choi 2005), and others examined the resonance induced by a mechanical oscillating boundary (e.g. Kim, Kim & Choi 2002).

Lage & Bejan (1993) and Kwak & Hyun (1996) numerically confirmed the occurrence of a cavity-wide resonance of natural convection triggered by a pulsating heat flux input and a time-dependent sinusoidal-like temperature boundary condition, respectively. Lage & Bejan (1993) also proposed a concept of a ‘fluid wheel’ to estimate the resonance frequency of natural convection in a differentially heated cavity.
By using the vertical velocity scale ($v$) of the natural convection boundary layer and the perimeter of the cavity ($P_{er}$), the resonance frequency was estimated to be $f \sim v/P_{er}$. For $Pr \geq 1$, the frequency scale is derived to be $f \sim 0.25Ra^{-1/10}Pr^{-1/2}$, and for $Pr < 1$, the scale is $f \sim 0.25(RaPr)^{-1/10}$. However, a comparison with their numerical results suggested that these scales were not able to predict the critical frequencies well, with up to 50% uncertainty (Lage & Bejan 1993). Based on the fluid wheel concept, Antohe & Lage (1996) and Kwak & Hyun (1996) proposed modified frequency scales by introducing more detailed velocity scales, such as the velocity scale of the horizontal intrusion. Kwak & Hyun (1996) further compared the numerical resonance frequency with the fluid wheel and internal wave-based (Paolucci & Chenoweth 1989) frequency scales. It turned out that the resonance frequency of the numerical results was very close to the frequency of the internal wave oscillations, whereas the predictions based on the fluid wheel resulted in quantitative discrepancies. This indicates that the observed resonance may be linked to the internal gravity wave oscillations, which is consistent with the argument of Kwak et al. (1998).

Another area of research motivated by enhancing heat transfer through triggering instabilities in natural convection boundary layers was explored by Xu, Patterson & Lei (2006, 2009a, 2011). It was demonstrated that a short non-metallic thin fin was able to trigger instabilities in the downstream boundary layer, which eventually resulted in $\sim 5\%$ heat transfer enhancement in the quasi-steady state (Xu, Patterson & Lei 2009b). The frequency characteristics of the boundary layers in the cases with and without a fin were also examined by Xu, Patterson & Lei (2010). It was found that the frequency characteristics of the temperature oscillations of the boundary layers were different for the cases with and without a fin. The general development of the boundary-layer frequency in the differentially heated cavity without a fin on the sidewall of interest was similar to that of a heated vertical flat plate, that is, the frequency band became narrower downstream, which may be due to the frequency-filtering effect of the boundary layer. In the case with a fin on the vertical wall, the frequency band downstream of the fin was widened, which was attributed to the occurrence of turbulence structures induced by the reattachment of the separated plumes to the downstream boundary layer (Xu et al. 2010).

The above literature survey demonstrates that the instability and resonance characteristics concerning natural convection boundary layers adjacent to a vertical isothermally heated plate have not been investigated thoroughly by previous linear stability analyses. This is due to the nature of the linear stability analysis, which is based on infinitesimal disturbances and provides only sufficient conditions for instability (McBain, Armfield & Desrayaud 2007). Furthermore, the existing understanding of the frequency characteristics of the vertical boundary-layer flows needs to be extended to exploit the resonance characteristics of the natural convection boundary layers in order to enhance heat transfer.

In this study, a direct stability analysis that solves the full set of perturbed nonlinear two-dimensional governing equations is adopted. The purpose of this study is two-fold: firstly, to extend the existing understanding of the evolution of the boundary-layer frequencies; and secondly, by seeding perturbations of various frequencies and finite amplitudes, to exploit the resonance characteristics of the thermal boundary layer and the effects of the resonance on heat transfer, particularly in the nonlinear flow response regime.

The remainder of the paper is organized as follows. Following the introduction, mathematical formulation and numerical procedures are given in § 2. Section 3 describes the investigation of the response of the boundary layer subjected to random
perturbations and the resonance behaviour of the boundary layer subjected to single-mode perturbations. In § 3.1 the frequency-filtering effect of the boundary layer is demonstrated. In § 3.2 evidence for the high-frequency band determining the resonance behaviour of the boundary-layer flow is presented. In § 3.3 the dependence of the characteristic frequency on the Rayleigh number is discussed, and in § 3.4 the frequency bands of resonance for $Ra$ in the range of $2.3 \times 10^8$ to $4.5 \times 10^9$ are determined. In §§ 3.5 and 3.6 the effects of resonance on the boundary-layer transition and heat transfer are described. Finally, major conclusions are presented in § 4.

2. Mathematical formulation and numerical procedures

2.1. Mathematical formulation

Under consideration is a Newtonian two-dimensional (2D) natural convection boundary-layer flow induced by an isothermally heated vertical plate of a dimensional height $H$ (refer to the schematic shown in figure 1). The width of the computational domain is $L_e = 0.5H$, which is determined based on the thickness scale of the viscous boundary layer (Patterson & Imberger 1980) to ensure that the far-field boundary condition is satisfied for the boundary-layer flow at the lowest Rayleigh number considered in this study. To minimize the effects of the lower horizontal boundary on the numerical solution, the vertical plate is extended downwards by $H_e$ ($H_e = 0.2H$ in all simulations), and an adiabatic and no-slip condition is assumed on the extended section. The same strategy was adopted in Lin, Armfield & Patterson (2008).

The flow in the considered domain is described by the 2D Navier–Stokes and energy equations, the non-dimensional form of which under the Boussinesq approximation can
be expressed as follows:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{Ra}{Pr} \theta,
\]

\[
\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{1}{Pr} \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) + S,
\]

where \( u \) and \( v \) are the velocity components in the \( x \) and \( y \) directions, respectively, \( p \) is the pressure, \( t \) is the time, \( \theta \) is the temperature, and \( S \) is a source term used to introduce perturbations (see the description at the end of this section). Here the quantities \( u, v, x, y, p, t \) and \( \theta \) are the dimensionless forms of the corresponding dimensional quantities \( U, V, X, Y, P, \tau \) and \( T \), respectively, with the following normalizations:

\[
\begin{align*}
\frac{u}{v \nu^{-1}}, \quad \frac{v}{\nu H^{-1}}, \quad \frac{x}{H}, \quad \frac{y}{H}, \\
\frac{t}{H^2 \nu^{-1}}, \quad \frac{p}{\rho \nu^2 H^{-2}}, \quad \frac{\theta}{T - T_0},
\end{align*}
\]

(2.5)

It is well known that the natural convection flow under consideration can be characterized by the Rayleigh and Prandtl numbers, which are defined as

\[
Ra = \frac{g \beta \Delta T H^3}{\nu \kappa}, \quad Pr = \frac{\nu}{\kappa},
\]

(2.6)

where \( g \) is the gravitational acceleration, \( \nu, \kappa \) and \( \beta \) are the kinematic viscosity, thermal diffusivity and thermal expansion coefficient of the working fluid at the reference ambient temperature \( T_0 \), \( H \) is the height of the heated surface and \( \Delta T = T_w - T_0 \) is the heating temperature difference.

The boundary conditions on the surface of interest are rigid no-slip and isothermal. The boundary conditions for the extended vertical surface and the adjacent horizontal surface are rigid no-slip and adiabatic. The top and right far-field boundaries are open where any backflow from the exterior is considered to be at the reference temperature. It is worth clarifying that the ‘end effect’ of the top and right far-field boundaries has been tested by comparing numerical results based on the computational domain (shown in figure 1) with and without extensions in the horizontal and vertical directions, respectively. For the top open boundary, the tests show that, by extending the computational domain from \( H \) to \( 1.5H \) (also see a similar approach in Lin et al. (2008)), subtle differences in the waveforms can be observed at \( y = 0.99 \), but they are not noticeable at \( y = 0.96 \). Therefore, the top open boundary exhibits a negligible effect on the properties of interest in this study, such as the response frequency, the temperature oscillation amplitude and the heat transfer rate of the quasi-steady stage flow. For the right far-field boundary, similarly by extending the computational domain from \( L_e \) to \( 1.5L_e \), it is found that the ‘end effect’ on the numerical results is indiscernible at all three flow stages (i.e. one-dimensional growth, transitional and quasi-steady stages). Based on the tests, no extension of the computational domain is necessary in the present study.
The fluid in the considered domain initially is stationary and isothermal at the temperature \( \theta = 0 \). The dimensionless forms of these initial conditions are expressed as

\[
\begin{align*}
    u = v = 0, & \quad \theta = 0 \quad \text{at } 0 \leq x \leq 0.5, \quad -0.2 \leq y \leq 1 \quad \text{for } t \leq 0.
\end{align*}
\] (2.7)

The boundary conditions described above can be written in dimensionless form as

\[
\begin{align*}
    u = v = 0, & \quad \theta = 1 \quad \text{at } x = 0, \quad 0 \leq y \leq 1, \\
    u = v = 0, & \quad \partial \theta / \partial x = 0 \quad \text{at } x = 0, \quad -0.2 \leq y < 0, \\
    u = v = 0, & \quad \partial \theta / \partial y = 0 \quad \text{at } 0 < x \leq 0.5, \quad y = -0.2, \\
    \partial u / \partial x = \partial v / \partial x = 0, & \quad \partial \theta / \partial x = 0 \quad \text{at } x = 0.5, \quad -0.2 < y < 1, \\
    \partial u / \partial y = \partial v / \partial y = 0, & \quad \partial \theta / \partial y = 0 \quad \text{at } 0 < x \leq 0.5, \quad y = 1,
\end{align*}
\] (2.8)

Artificial perturbations are introduced through the source term \( S \) in (2.4) into the boundary-layer flow over a small region at the base of the boundary layer \( (0 \leq x \leq 0.02, \quad 0 \leq y \leq 0.02) \), as shown in figure 1. The perturbations are evaluated at each time step and applied uniformly across the perturbed region. Two modes of perturbations, i.e. random- and single-mode perturbations, are applied in this study. For the random-mode perturbation experiments, the source term \( S \) is specified as: \( S = 2A \text{rand}(0, 1) - 0.5 \), with \( A = 9 \), where \( \text{rand}(0, 1) \) is a random number generator, generating statistically uniformly distributed random numbers between 0 and 1. Here the perturbation amplitude \( A \) is determined by referring to the perturbation amplitude adopted in Armfield & Janssen (1996). As the thermal boundary-layer flow considered in Armfield & Janssen (1996) is in the context of an enclosed cavity, which is different from the present investigation, which has no ambient thermal stratification, the perturbation amplitude \( A \) has been further tested to ensure that the response of the thermal boundary layer remains in the linear response regime (refer to § 2.2). For the single-mode perturbation experiments, the source term is specified as \( S = A_s \sin(2\pi f_{pt} t) \). Here \( f_{pt} \) is the perturbation frequency and \( A_s \) is the perturbation amplitude \( (A_s = 3.6A, \quad \text{unless specified otherwise}) \). The suitability of using artificially introduced energy perturbations to investigate the instability of natural convection boundary layers has been discussed and verified in Janssen & Armfield (1996).

### 2.2. Numerical procedures and tests

The governing equations along with the initial and boundary conditions are solved implicitly using a finite-volume method with the SIMPLE scheme (see Patankar 1980) for pressure–velocity coupling. The spatial derivatives are discretized using the second-order central-differencing scheme, except for the advection terms, which are approximated by the QUICK scheme (see Leonard 1979). The unsteady terms are integrated by a second-order backward difference scheme.

The dependences of the numerical results on mesh, time step and perturbation amplitude are first tested for the cases with random-mode perturbations at \( Ra = 4.5 \times 10^9 \) and \( Pr = 7 \). The dependences on time step and mesh are also conducted for the same flow condition (i.e. the same \( Ra \) and \( Pr \) but with single-mode perturbations. The perturbation frequency applied in the single-mode perturbation tests is the characteristic frequency of the boundary layer, the determination of which will be described in §§ 3.1 and 3.2.

With the random-mode perturbation tests, a mesh-dependence test is first conducted on three sets of non-uniform meshes, i.e. \( 400 \times 149, \quad 700 \times 249 \) and \( 800 \times 299 \). The tested grid systems are all constructed with concentrated grids in the proximity of
vertical wall boundaries. The grid size next to the vertical wall is $\Delta x_w = 8 \times 10^{-4}$ and the grid has a 0.3–1% linear stretching in the horizontal ($x$) direction. The grid is uniform in the vertical direction for all the tested meshes, which gives $\Delta y = 1.7 \times 10^{-3}$ in the medium grid system. The power spectra of the temperature time series at $x = 6.25 \times 10^{-3}$ and $y = 0.5$ obtained with the three different meshes are shown in figure 2(a). Note that $P(f)$ is the spectral power, which is obtained by performing a fast Fourier transform of the temperature time series. Here the power spectra have been smoothed using a 15-point moving average in order to discern their main features. Figure 2(a) demonstrates that the two finer mesh systems $700 \times 249$ and $800 \times 299$ produce very similar results, whereas the coarsest mesh system $400 \times 149$ produces discernible variations. A quantitative comparison of the influence of these three mesh systems on the results is shown in table 1. Here the peak frequency and the root-mean-square (r.m.s.) value of the temperature time series are obtained at the same monitored position (i.e. $x = 6.25 \times 10^{-3}$, $y = 0.5$) over the time period $0.0105–0.0175$. The comparison of cases 1, 4 and 5 indicates that the $700 \times 249$ mesh is able to provide sufficient spatial resolution. The medium mesh, i.e. $700 \times 249$, is therefore chosen for subsequent direct stability analyses in consideration of computing costs.

For the time step-dependence test, three time steps ($4\Delta t$, $\Delta t$ and $\Delta t/2$) are tested with the adopted mesh $700 \times 249$. Here the non-dimensional time step $\Delta t$ is $1.7 \times 10^{-7}$. The power spectra of the temperature at the same monitoring position obtained with these three time steps are shown in figure 2(b). It can be seen that the spectra profiles based on the time steps $\Delta t$ and $\Delta t/2$ are very similar. The quantitative comparison of the peak frequency and r.m.s. of the temperature time series between cases 1 and 6 in table 1 suggests that the medium time step $\Delta t$ is adequate, and thus it is adopted in this study. It is worth clarifying that the non-monotonic change of the obtained peak frequency with reducing time step may be attributed to the different lengths of the data points for performing fast Fourier transform, since the same physical time period at the quasi-steady stage is used for data processing. The peak frequency of case 7 may also be distorted because of the smaller ensemble size.

Finally, the effect of the amplitude of the perturbations is tested to ensure the boundary-layer response is in the linear response regime. Figure 2(c) shows the power spectra obtained with three different perturbation amplitudes, i.e. $A_{pt} = 2A$, $A$ and $A/2$. It is worth noting that the spectral power is normalized by the square of the perturbation amplitude in order to account for the different perturbation amplitudes. It is seen in figure 2(c) that the normalized spectral profile and the peak frequency region in the frequency range of interest (i.e. from 0 to $7 \times 10^4$) are almost independent of the perturbation amplitude, confirming that the boundary layer is indeed in the linear response regime within the present range of perturbation amplitudes. The r.m.s. values of the temperature time series obtained with these three amplitudes, i.e. cases 1, 2 and 3 in table 1, further confirm that the response of the boundary layer is in the linear response regime. Similar results are obtained for other monitoring positions. In the subsequent calculations, the amplitude $A$ is thus adopted for all random-mode perturbation experiments.

For the single-mode perturbation tests, the dependences of the numerical results on the time step and mesh are tested using the same time steps and meshes as those used for the above-described random-mode perturbation tests. Quantitative comparisons of the results obtained at the same monitored position over the same time period but with different meshes and time steps are shown in table 1. The comparison of the temperature standard deviation in cases 8, 9 and 10 suggests that the time step $\Delta t$ is able to give sufficient resolution and thus is adopted. The comparison of cases
<table>
<thead>
<tr>
<th>Case no.</th>
<th>Perturbation mode</th>
<th>Mesh</th>
<th>Time step</th>
<th>Perturbation amplitude</th>
<th>Peak frequency</th>
<th>R.m.s. of $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Random</td>
<td>$700 \times 249$</td>
<td>$\Delta t$</td>
<td>$A$</td>
<td>37 898</td>
<td>$1.7 \times 10^{-4}$</td>
</tr>
<tr>
<td>2</td>
<td>Random</td>
<td>$700 \times 249$</td>
<td>$\Delta t$</td>
<td>$2A$</td>
<td>$37 898 \times 1.02$</td>
<td>$1.7 \times 10^{-4} \times 1.94$</td>
</tr>
<tr>
<td>3</td>
<td>Random</td>
<td>$700 \times 249$</td>
<td>$\Delta t$</td>
<td>$A/2$</td>
<td>$37 898 \times 1.03$</td>
<td>$1.7 \times 10^{-4} \times 0.52$</td>
</tr>
<tr>
<td>4</td>
<td>Random</td>
<td>$400 \times 149$</td>
<td>$\Delta t$</td>
<td>$A$</td>
<td>$37 898 \times 1.10$</td>
<td>$1.7 \times 10^{-4} \times 0.71$</td>
</tr>
<tr>
<td>5</td>
<td>Random</td>
<td>$800 \times 299$</td>
<td>$\Delta t$</td>
<td>$A$</td>
<td>$37 898 \times 0.97$</td>
<td>$1.7 \times 10^{-4} \times 1.06$</td>
</tr>
<tr>
<td>6</td>
<td>Random</td>
<td>$700 \times 249$</td>
<td>$\Delta t/2$</td>
<td>$A$</td>
<td>$37 898 \times 1.05$</td>
<td>$1.7 \times 10^{-4} \times 0.88$</td>
</tr>
<tr>
<td>7</td>
<td>Random</td>
<td>$700 \times 249$</td>
<td>$4\Delta t$</td>
<td>$A$</td>
<td>$37 898 \times 1.20$</td>
<td>$1.7 \times 10^{-4} \times 1.82$</td>
</tr>
<tr>
<td>8</td>
<td>Single ($f_{pt} = f_c$)</td>
<td>$700 \times 249$</td>
<td>$\Delta t$</td>
<td>$A_s$</td>
<td>—</td>
<td>$9.9 \times 10^{-3}$</td>
</tr>
<tr>
<td>9</td>
<td>Single ($f_{pt} = f_c$)</td>
<td>$700 \times 249$</td>
<td>$\Delta t/2$</td>
<td>$A_s$</td>
<td>—</td>
<td>$9.9 \times 10^{-3} \times 1.00$</td>
</tr>
<tr>
<td>10</td>
<td>Single ($f_{pt} = f_c$)</td>
<td>$700 \times 249$</td>
<td>$4\Delta t$</td>
<td>$A_s$</td>
<td>—</td>
<td>$9.9 \times 10^{-3} \times 0.96$</td>
</tr>
<tr>
<td>11</td>
<td>Single ($f_{pt} = f_c$)</td>
<td>$400 \times 149$</td>
<td>$\Delta t$</td>
<td>$A_s$</td>
<td>—</td>
<td>$9.9 \times 10^{-3} \times 0.98$</td>
</tr>
<tr>
<td>12</td>
<td>Single ($f_{pt} = f_c$)</td>
<td>$800 \times 299$</td>
<td>$\Delta t$</td>
<td>$A_s$</td>
<td>—</td>
<td>$9.9 \times 10^{-3} \times 1.00$</td>
</tr>
</tbody>
</table>

**Table 1.** Peak frequency and root-mean-square (r.m.s.) value of the temperature time series. Data presented here are all obtained at the monitored position $(x, y) = (6.25 \times 10^{-3}, 0.5)$ over the time period 0.0105–0.0175. All cases are for $Ra = 4.5 \times 10^9$, $Pr = 7$ and $\Delta t = 1.7 \times 10^{-7}$. Here $A = 9$ and $A_s = 3.6A$. 
Figure 2. (Colour online) (a) Mesh, (b) time step and (c) perturbation-amplitude dependence tests for $Ra = 4.5 \times 10^9$ and $Pr = 7$. Plotted are the power spectra of the temperature (at $x = 6.25 \times 10^{-3}, y = 0.5$) in the thermal boundary layer.
8, 11 and 12 indicates that the influence of these three meshes on the numerical results is insignificant. The medium mesh is therefore also chosen for the subsequent calculations with single-mode perturbations.

3. Results and discussion

3.1. Response to random-mode perturbations

In order to obtain insights into the frequency-filtering effect and frequency evolution behaviour of the thermal boundary layer, a direct stability analysis is performed first for $Ra = 4.5 \times 10^9$ and $Pr = 7$, with random perturbations superimposed onto the base flow near the leading edge of the boundary layer. The calculation is extended until the thermal boundary layer reaches a quasi-steady (or statistically steady) stage.

The temperature time series obtained in the thermal boundary layer at two representative positions in the thermal boundary layer are shown in figure 3(a) to illustrate the overall temperature evolution under random perturbations. It is seen that the overall evolution can be broadly classified into three stages: one-dimensional growth stage, transitional stage and quasi-steady stage. The first two stages are the same as those discussed in Patterson, Graham & Schöpf (2002). The third stage is slightly different from the steady stage discussed in Patterson et al. (2002). The insets in figure 3(a) indicate that the thermal boundary layer subjected to random perturbations finally becomes oscillatory, presenting a certain inherent oscillation frequency, i.e. becoming quasi-steady or statistically steady. The analyses presented below are based on the data extracted at the quasi-steady stage. The same procedure is then carried out for an additional four Rayleigh numbers to examine the dependence of the characteristic frequencies on the Rayleigh numbers, which will be described in § 3.3.

The temperature time series obtained in the thermal boundary layer over a range of vertical positions at the quasi-steady stage are shown in figure 3(b). Figure 3(c) reproduces part of the data plotted in figure 3(b), at a finer scale, in order to demonstrate the detail of the oscillatory behaviour at the quasi-steady stage. All the temperature time series in the present work are taken at a distance of $x = 6.25 \times 10^{-3}$ from the vertical wall unless specified otherwise. It is worth noting that, despite the different temperature ranges shown for various locations in figure 3(b), the same temperature difference scales are used in order to discern the variation of the temperature fluctuation at different streamwise positions. Similarly, the temperature difference scales for the plots of $y = 0.04$, 0.17 and 0.33 in figure 3(c) are the same.

The comparison of the temperature time series at $y = 0.04$ and 0.17 in figure 3(c) indicates that the perturbations are decayed in the very upstream section of the boundary layer, which will be examined in more detail later. As the perturbations are convected further downstream, they are amplified by the thermal boundary layer, as indicated by the changing amplitudes of the temperature time series at $y = 0.33$, 0.50, 0.67 and 0.83, respectively. The initially decaying and then amplifying behaviour of the perturbations in the thermal boundary layer is a common feature observed for all Rayleigh numbers. Furthermore, it is seen in figure 3(c) that the temperature oscillation in the upstream boundary layer is relatively irregular, whereas it becomes increasingly regular in the downstream boundary layer. This implies that the random perturbations introduced at the upstream end are being filtered by the thermal boundary layer as they are propagating downstream.

Spectral analyses of the temperature time series at selected streamwise positions over $0.014 \leq t \leq 0.017$ are then carried out to examine the frequency-filtering effect of
the boundary layer. Figure 4 presents the spatial evolution of the power spectra of the temperature time series in the streamwise direction (from bottom to top). It is clear in figure 4 that the complete boundary layer may be divided into three distinct regions, which includes an upstream low-frequency region, a transitional region (with both low and high frequencies) and a downstream high-frequency region. It is worth noting that the random perturbations introduced upstream are similar to white noise, which contains a full range of frequencies. It is seen in figure 4(a) that only a narrow band of low frequencies of the random perturbations has survived in the upstream boundary layer. As the perturbations are convected downstream, another high-frequency band appears in the spectra and grows in the streamwise direction, while the low-frequency band decays (indicated by the variation of the power of the two bands shown in Figure 3. (Colour online) Temperature time series at different vertical positions in the thermal boundary layer obtained for \( Ra = 4.5 \times 10^9 \) and \( Pr = 7 \). (a) Overall temperature evolution obtained at \( y = 0.17 \) and 0.83, respectively; (b) temperature time series over the time period 0.0105–0.0175 at various heights; and (c) temperature time series over the time period 0.0145–0.0155, showing details of the temperature oscillations. The dimensionless temperatures have been multiplied by 100 for clarity. The y scales for temperature time series in panels (b) and (c) are arranged alternately on the left and right sides for clarity.
Figure 4. (Colour online) Power spectra of the temperature time series at various streamwise locations, demonstrating the evolution of the boundary layer frequencies. Results are for the case with $Ra = 4.5 \times 10^9$ and $Pr = 7$. (a) The upstream low-frequency region; (b)–(d) the transitional region with both the low- and high-frequency bands; (e) the downstream high-frequency region.
The peak frequencies $f_p$ (with the highest power on the spectra) at different streamwise locations of the boundary layer are also indicated in figure 4. It is seen in figure 4(a–c) that, in the upstream part of the boundary layer, the peak frequency initially appears in the low-frequency band. Further downstream, associated with the decay of the low-frequency band and the growth of the high-frequency band, the peak frequency appears in the high-frequency band (i.e. the characteristic frequency band of the thermal boundary layer), as shown in figure 4(d,e). It is also observed that, above the height $y = 0.25$, the peak frequency remains constant, which is therefore referred to as the characteristic frequency of the thermal boundary layer (refer to Gebhart & Mahajan 1975). The spatial evolution of the peak frequency $f_p$ and its corresponding power $P(f)$ in the streamwise direction are plotted in figure 5. It is interesting to note that the evolution of the peak frequency appears as a three-
step development, approximately corresponding to the three distinct regions described above. Figure 5(a) confirms that the characteristic frequency appears around $y = 0.25$, above which the peak frequency remains constant. The dimensionless characteristic frequency determined by this random perturbation experiment is 37898 for the present case ($Ra = 4.5 \times 10^9$ and $Pr = 7$). It is also seen in figure 5(b) that the spatial evolution of the spectral power of the peak frequency includes two stages, firstly a decaying stage upstream and secondly an amplifying stage downstream. The decaying stage is confined in a small upstream region, which covers the low-frequency region and some of the transitional region described above.

3.2. Response to single-mode perturbations

In order to confirm the frequency characteristics observed in the random-mode perturbation experiment described above and to obtain further insights into the resonance characteristics of the thermal boundary layer, single-mode perturbation experiments over a range of perturbation frequencies are performed at the same Rayleigh and Prandtl numbers as considered in § 3.1 (i.e. $Ra = 4.5 \times 10^9$ and $Pr = 7$).

In the single-mode perturbation experiments, the perturbation frequency $f_{pt}$ starts at the characteristic frequency $f_c$ identified above and then varies in both directions (i.e. increasing and decreasing from $f_c$).

Figure 6(a) shows the temperature time series at various streamwise locations of the thermal boundary layer perturbed at its characteristic frequency, that is, $f_{pt} = f_c$. It is clear in this figure that the temperature time series appear in the form of sinusoidal-like oscillations except in the very upstream region where the thermal boundary-layer flow is stable. The stable region approximately corresponds to the decaying region shown in figure 5(b) and is due to the very low local Rayleigh numbers (defined in terms of the distance from the leading edge). It can also be seen in figure 6(a) that the sinusoidal-like oscillations are significantly amplified in the
Figure 7. (Colour online) Temperature time series at $y = 0.83$ in the boundary layer perturbed at different perturbation frequencies ($Ra = 4.5 \times 10^9$ and $Pr = 7$). The difference between two consecutive perturbation frequencies is 8600. The temperature time series are shifted vertically for clarity. The shifted values correspond to the temperature readings at $t = 0$.

Streamwise direction, especially in the downstream region where the thermal boundary layer is more unstable because of the relatively higher local Rayleigh numbers.

Figure 6(b) compares the temperature time series at $y = 0.58$ for the unperturbed (upper curve) and the perturbed (lower curve) boundary-layer flows. Clearly, both time series show three stages of flow development, similar to that described in Patterson et al. (2002). The first and second stages are classified respectively as the one-dimensional growth stage ($S_C$) dominated by conduction and the transitional stage ($S_T$) characterized by the arrival of the leading-edge effect and subsequent occurrence of travelling waves. Both the perturbed and unperturbed thermal boundary layers show similar behaviour in these two stages. In the third stage, the unperturbed thermal boundary-layer flow is steady, referred to as the steady stage ($S_S$) in the previous studies. For the perturbed case, however, the temperature time series undergoes periodic oscillations due to the presence of the single-mode perturbations. The same comparative features are observed at other locations between the perturbed and unperturbed thermal boundary layers. The third stage is referred to as the quasi-steady stage ($S_{QS}$) in this study. Subsequent discussions are mostly concerned with the boundary-layer properties at the quasi-steady stage.

The response of the thermal boundary layer to single-mode perturbations of different perturbation frequencies is demonstrated in figure 7. The temperature time series in this figure are all obtained at $y = 0.83$. It is seen that the maximum oscillation amplitude at the quasi-steady stage is triggered by the perturbation at the characteristic frequency $f_c$. As the perturbation frequency deviates from the characteristic frequency in either direction, the amplitude of the temperature signals decreases. This result clearly demonstrates that the boundary layer responds strongly to a particular band of frequencies, which may be referred to as the frequency band of resonance $B_r$.

The vertical temperature profiles at $t = 6 \times 10^{-4}$ in the thermal boundary layers perturbed at different perturbation frequencies are shown in figure 8, in which the temperature is measured along the vertical line of $x = 6.25 \times 10^{-3}$ and the interval of
the perturbation frequency $\Delta f$ is 8600. Figure 8 clearly confirms that the perturbation at the characteristic frequency results in the largest travelling wave amplitude in the downstream boundary layer, whereas perturbations at frequencies deviated from the characteristic frequency result in much smaller wave amplitudes. The temperature fluctuation becomes much smaller at $f_{pt} = f_c + 3\Delta f$ and $f_{pt} = f_c - 3\Delta f$, suggesting that the boundary layer responds appreciably to a frequency band approximately between $f_c - 3\Delta f$ and $f_c + 3\Delta f$. It is also seen that the oscillations of the temperatures are indiscernible in the upstream part of the boundary layer. However, the perturbations start to grow appreciably above a certain height and are amplified rapidly in the streamwise direction.

In order to quantify the spatial growth of the perturbation, the oscillation amplitude of the temperature time series in the quasi-steady stage for the case $Ra = 4.5 \times 10^9$ and $Pr = 7$ perturbed at its characteristic frequency is plotted as a function of the height in figure 9. Similar profiles are also obtained at other perturbation frequencies. The growth curve shown in figure 9 can be represented by the exponential function

$$\psi = ae^{cy} \quad (y \geq 0),$$

where $\psi$ is the oscillation amplitude of the temperature time series at the quasi-steady stage, $a$ is an initial amplitude, $c$ represents the spatial growth rate of the oscillation amplitude and $y$ is the position in the streamwise direction. The exponential curve fitting shown in figure 9 is obtained with an adjusted $R^2$ of 0.996.

Following the same curve fitting procedure, the spatial growth rate of the boundary-layer flow at $Ra = 4.5 \times 10^9$ and $Pr = 7$ subject to single-mode perturbations of various frequencies is obtained. The effect of the perturbation frequency on the spatial growth rate of the perturbation is shown in figure 10. It is seen in figure 10 that the spatial growth rate reduces sharply as the perturbation frequency moves away from the
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**Figure 9.** (Colour online) Variation of the temperature oscillation amplitude $\psi$ in the streamwise direction ($Ra = 4.5 \times 10^9$, $Pr = 7$, $f_{pt} = f_c$).

**Figure 10.** (Colour online) The effects of the perturbation frequency on the spatial growth rate $c$ of perturbations ($Ra = 4.5 \times 10^9$, $Pr = 7$).

characteristic frequency. This further confirms that the boundary layer only strongly amplifies perturbations in a particular narrow band of frequencies. Similar behaviour has also been experimentally observed for a thermal boundary layer adjacent to an isoflux surface (refer to Polymeropoulos & Gebhart 1967).

In order to determine the cut-off frequencies of the amplification frequency band, the above-described perturbation frequency range is extended in both directions. Figure 11 plots the square of the temperature fluctuation amplitudes ($\psi^2$) monitored at an upstream ($y = 0.04$) and a downstream ($y = 0.83$) position, respectively, in
the thermal boundary layer against the perturbation frequency. The amplification or otherwise of perturbations at a particular frequency can be seen clearly by comparing the fluctuation amplitudes at the two locations. The intersections of the two curves are thus identified as the cut-off frequencies. Accordingly, the lower cut-off frequency is determined to be $f_{LC} = 0.97 \times 10^4$, and the upper cut-off frequency is determined to be $f_{UC} = 9.75 \times 10^4$. This result suggests that a perturbation will be amplified by the thermal boundary layer if its frequency is within the amplification frequency band (i.e. $f_{LC} < f < f_{UC}$).

3.3. Dependence of the characteristic frequency on the Rayleigh number

The above-described random perturbation experiments are also carried out for four additional Rayleigh numbers (i.e. $Ra = 2.3 \times 10^8$, $6.0 \times 10^8$, $1.8 \times 10^9$ and $3.4 \times 10^9$, respectively) to examine the dependence of the characteristic frequency on the Rayleigh number. Figure 12(a) illustrates the spatial evolution of the peak frequency for all five Rayleigh numbers calculated in this study. The dimensionless characteristic frequencies determined at these five Rayleigh numbers are 4881, 8326, 20672, 29572 and 37898, respectively, which generally agree with the results predicted by the semi-analytical estimation $f' = Bu^{-1/3} (g\beta\Delta T)^{2/3} (2\pi)^{-1}$ proposed by Gebhart & Mahajan (1975) with variations between 3.4 and 13.9%. Note that the frequencies obtained from the semi-analytical estimation have been normalized before comparison. The variation between the present and previous investigations may be partly due to the fact that the semi-analytical estimation includes an empirical correlation of $B = 0.315Pr^{-0.065}$ based on experimental data with a 10% uncertainty. The dependence of the critical height (denoted by $y_c$), at which the characteristic frequency first becomes dominant, on the Rayleigh number is plotted in figure 12(b). It is clear in this plot that the critical height reduces (moves upstream) as $Ra$ increases, indicating that the frequency filtering process in the thermal boundary layer finishes sooner at higher Rayleigh numbers.

The dependence of the characteristic frequency on the Rayleigh number is shown in figure 12(c). Since the natural convection boundary-layer flows are characterized by the Prandtl and Rayleigh numbers, the functional form describing the dependence of the characteristic frequency on the controlling parameters may be represented by $f_c = F(Pr, Ra)$. Considering $Pr$ as a constant ($Pr = 7$) in the present study, a correlation between the characteristic frequency $f_c$ and the Rayleigh number $Ra$ is
Figure 12. (Colour online) (a) Streamwise evaluation of the peak frequency at $Ra$ over the range of $2.3 \times 10^8$ to $4.5 \times 10^9$; (b) the dependence of the critical height $y_c$ on the Rayleigh number over $2.3 \times 10^8$ to $4.5 \times 10^9$; and (c) the dependence of the characteristic frequency $f_c$ on the Rayleigh number over $2.3 \times 10^8$ to $3.6 \times 10^9$. In panel (c), the solid line represents the fitted curve for the five characteristic frequencies (marked with solid squares) determined by direct stability analyses; the dash-dotted line is an extension of the curve fitting represented by $f_c = 0.0136Ra^{2/3}$; the dashed line is from the estimation of Gebhart & Mahajan (1975).
obtained as follows, with an adjusted $R^2$ of 0.995:

$$f_c = 0.0136 Ra^{2/3}. \quad (3.2)$$

An additional random perturbation experiment is performed for a case at $Ra = 3.6 \times 10^{10}$ to further examine the applicability of the above frequency correlation to higher Rayleigh numbers. The difference between the characteristic frequencies determined from (3.2) and that from the random perturbation experiment is only 2%, which indicates that the above correlation may be applied to Rayleigh numbers up to $3.6 \times 10^{10}$. The hollow square in figure 12(c) shows the characteristic frequency determined by direct stability analysis for the flow at $Ra = 3.6 \times 10^{10}$. Over the Rayleigh number range of $2.3 \times 10^8$ to $3.6 \times 10^{10}$, the power-law scaling is compared with the estimation of Gebhart & Mahajan (1975), as demonstrated in figure 12(c). It is seen that Gebhart and Mahajan’s estimation predicts the characteristic frequencies well at relatively low Rayleigh numbers, whereas it slightly underpredicts the characteristic frequencies at relatively high Rayleigh numbers.

It is worth clarifying that the numerical simulation for the case of $Ra = 3.6 \times 10^{10}$ is performed by doubling the length of the heated plate used for $Ra = 4.5 \times 10^9$ while keeping all other parameter settings the same. The mesh for the extended section of the computational domain is constructed with an equivalent resolution to the mesh adopted for $Ra = 4.5 \times 10^9$, and thus is considered to be adequate.

The above power-law scaling of the characteristic frequency with the Rayleigh number (i.e. equation (3.2)) can be further interpreted by converting it into a dimensional form, which is obtained by multiplying both sides of the equation by the frequency scale $(1/H^2 \nu^{-1})$, which gives

$$f^*_c \sim \left(\frac{g \beta \Delta T}{\kappa} \right)^{2/3} \nu^{1/3}, \quad (3.3)$$

where $f^*_c$ denotes the dimensional characteristic frequency. It is clear from (3.3) that the dimensional characteristic frequency is independent of the height of the heated surface. This physical interpretation is consistent with the observation of Gebhart & Mahajan (1975).

### 3.4. Frequency bands of resonance

The numerical evidence presented in § 3.2 has clearly demonstrated that the thermal boundary layer favours a particular band of perturbation frequencies for amplification. In this section, we further identify a sub-band of frequencies within which the perturbations are strongly amplified. This sub-band is referred to as the resonance frequency band, which will be determined approximately for the five Rayleigh numbers over the range of $2.3 \times 10^8$ to $4.5 \times 10^9$.

Figure 13 plots the standard deviation of the temperature oscillations against the frequency of the single-mode perturbations at three different locations for $Ra = 4.5 \times 10^9$ and $Pr = 7$. The power spectrum of the temperature time series obtained at $y = 0.83$ from the random perturbation experiment described above is also superimposed on the plot. As expected, the maximum standard deviation is observed at all three locations at a perturbation frequency around 38000, which is the same as the characteristic frequency of the thermal boundary layer identified above. Figure 13 also shows excellent correlation between the random-mode and single-mode perturbation experiments in terms of the frequency response. Both numerical experiments suggest that the lower bound of the resonance frequency band $f_{LR}$ is $\sim 2 \times 10^4$ and the upper
Figure 13. (Colour online) Comparison of the frequency bands obtained from the random- and single-mode perturbation experiments for \(Ra = 4.5 \times 10^9\) and \(Pr = 7.0\).

Figure 14. (Colour online) Determination of the lower and upper bounds of the resonance frequency band and illustration of the various frequency bands across the tested frequency range: DFB, decay frequency band; AFB, amplification frequency band; WAFB, weak amplification frequency band; RFB, resonance frequency band (i.e. strong amplification frequency band). The plot is for \(Ra = 4.5 \times 10^9\) and \(Pr = 7.0\).

Bound \(f_{UR}\) is \(\sim 7 \times 10^4\). This result implies that the resonance frequency band of a thermal boundary layer at a given Rayleigh number can be determined by performing either random- or single-mode perturbation experiments. The characteristic frequency band of the thermal boundary layer (i.e. the surviving high-frequency band shown in figure 4e) determines the resonance properties of the thermal boundary layer.

Figure 14 plots the square of the amplitude of temperature fluctuations obtained at \(y = 0.83\) at the quasi-steady stage against the perturbation frequency for \(Ra = 4.5 \times 10^9\) and \(Pr = 7.0\). The amplification frequency band (AFB) determined in §3.2 is identified in this figure. Clearly, the AFB can be further divided into two sub-bands, an outer band in which perturbations are weakly amplified (i.e. WAFB – weakly amplified frequency band) and an inner band in which perturbations are strongly amplified (i.e.
RFB – resonance frequency band). There is a smooth transition between the WAFB and the RFB. Therefore, it is difficult to precisely determine the cut-off frequencies of the RFB. In this study, the lower and upper bounds of the RFB are determined approximately in figure 14. Here the response curve on either side of the characteristic frequency in the RFB is fitted with a linear line. The intersections of the linearly fitted curves with the horizontal line of $\psi^2 = 0$ are regarded as the lower and upper bounds of the RFB, respectively. For the given parameter settings, the lower bound of the RFB is determined to be $f_{LR} = 25,500$ and the upper bound is determined to be $f_{UR} = 55,200$. Since any perturbation within the RFB is strongly amplified by the thermal boundary layer, the identification of the RFB has significant implication for the enhancement of heat transfer (refer to § 3.6).

Figure 15 shows the resonance frequency bands (corresponding to the strong amplification frequency band) for $Ra$ over the range of $2.3 \times 10^8$ to $4.5 \times 10^9$, determined using the approach described above. It is clear in this figure that the resonance frequency band becomes wider as $Ra$ increases, indicating that the boundary-layer flow at the lower Rayleigh numbers is more selective of perturbation frequencies for amplification.

**3.5. The effects of resonance on boundary-layer transition**

This section examines the effects of the resonance on the laminar–turbulent transition of the thermal boundary layer. Numerical experiments are performed at $Ra = 3.6 \times 10^{10}$. First the characteristic frequency of the thermal boundary layer at this Rayleigh number is determined through a random perturbation experiment, which gives the characteristic frequency of $1.516 \times 10^5$. Subsequently the thermal boundary layer is perturbed by single-mode perturbations at the characteristic frequency. The results are described below.

Figure 16(a) compares the temperature time series at $y = 0.99$ for the perturbed ($f_{pt} = f_c$) and unperturbed boundary layers. Similar to the behaviour observed in figure 6(b), the time series of the temperature in the perturbed and unperturbed thermal boundary layers at $Ra = 3.6 \times 10^{10}$ show similar behaviour in the initial and transitional
Figure 16. (Colour online) (a) Comparison of the temperature time series in the perturbed and unperturbed boundary layers for $Ra = 3.6 \times 10^{10}$. The insets show the details of temperature oscillation at the start and a later stage of the quasi-steady stage for the perturbed case. (b,c) Power spectra of the temperature time series over $5 \times 10^{-4} \leq t \leq 10^{-3}$ and $2.5 \times 10^{-3} \leq t \leq 3 \times 10^{-3}$, respectively, for the perturbed case.

stages, but different features in the quasi-steady stage. It is worth noting that, in the quasi-steady stage, the temperature of the perturbed boundary layer at $y = 0.99$ does not oscillate about the mean temperature of the unperturbed case. This is caused by the nonlinearity of the perturbed boundary-layer flow, which has proceeded to a nonlinear response regime at $y = 0.99$. When the flow is still in the linear response regime (for example, in the boundary layer further upstream), the temperature of the perturbed boundary layer oscillates about the mean temperature of the corresponding unperturbed case. Further evidence for the transition from the linear to the nonlinear response regime is discussed below in detail.

Figure 16(b,c) shows the power spectra of the perturbed temperature time series over the time intervals of $5 \times 10^{-4} \leq t \leq 10^{-3}$ and $2.5 \times 10^{-3} \leq t \leq 3 \times 10^{-3}$, respectively. The obtained spectra are almost identical, confirming that the thermal boundary has indeed reached a quasi-steady stage after $t = 5 \times 10^{-4}$. It is interesting to note that a second peak appears in the spectra of figure 16(b,c), corresponding to the exact higher harmonic of the fundamental frequency. The presence of the higher harmonic component may be associated with a period-doubling bifurcation, indicating that the flow is undergoing a certain transition.

Figure 17(a) depicts the temperature time series at selected streamwise locations at the quasi-steady state. It is seen that the temperature oscillations are approximately sinusoidal for all locations below $y = 0.29$. Further downstream the sinusoidal behaviour becomes distorted and multiple frequency components start to appear, which may indicate the start of transition to the nonlinear regime. Similar behaviour was also experimentally observed during the transition of the thermal boundary layer adjacent to a uniformly heated vertical surface (see Qureshi & Gebhart 1978).

In order to examine the above-mentioned distortion process closely, the corresponding phase trajectories of the temperature time series shown in figure 17(a) are presented in figure 17(b). Note that the quantity $\partial \theta / \partial t$ is calculated using a
Resonance of thermal boundary layer adjacent to heated surface

Figure 17. (Colour online) (a) Temperature time series at selected heights in the thermal boundary layer perturbed at the characteristic frequency \((Ra = 3.6 \times 10^{10})\). The dimensionless temperatures have been multiplied by a factor of 100 for clarity. (b) Limit cycles of the phase trajectories of the corresponding temperatures time series shown in (a). The square insets indicate the presence of small loops formed in the phase trajectories, and a close-up of a typical small loop is shown in panel (iii).

The central difference approximation with respect to time. The scales of \(\partial \theta/\partial t\) and \(\theta\) are not relevant to the discussion and thus not shown in figure 17(b). It is seen in figure 17(b i) that the phase trajectory of the temperature time series at \(y = 0.10\) is close to an ellipse, suggesting that the wave of the temperature disturbance at this position is approximately sinusoidal, which can be further interpreted as that the thermal boundary-layer flow at this position is still in the linear growth regime. As \(y\) increases to \(y = 0.29\), the ellipsoidal phase trajectory has changed slightly, with its left portion becoming sharp, which indicates that the nonlinearity is developing, but at an extremely low rate. At this position, the linearity still dominates the flow. Further downstream to \(y = 0.58, 0.60\) and \(0.75\) (i.e. figure 17b iii–v), the initial ellipsoidal phase trajectory has distorted considerably. It is worth noting that a small loop has formed in each phase trajectory, as highlighted in the square insets and the close-up of a typical loop in figure 17(b iii). The appearance of these small loops in these phase trajectories suggests that the nonlinearity has developed to an appreciable level and the boundary-layer flow contains another high-frequency signal. In the far downstream
In order to quantify the position of transition from the linear to nonlinear response regime in the thermal boundary layer, the streamwise profiles of the standard deviation of the temperatures in the thermal boundary layer and the local Nusselt number are further examined, as shown in figure 18.

Figure 18(a) presents the calculated standard deviation of the temperatures at the quasi-steady stage at all monitored positions. It can be seen that the standard deviation of the temperatures initially undergoes smooth and rapid growth (up to $y = 0.58$), retains a peak value at $y = 0.58$, and then fluctuates irregularly. The irregularity of the standard deviation profile in the downstream section also suggests the transition to the nonlinear regime. It is worth noting that the critical height at which the standard deviation peaks approximately corresponds to the location where

region, as shown in figure 17(b vi), the small loop disappears and the phase trajectory is far from an ellipse, which is due to the further development and growth of the nonlinearity. It is worth pointing out that the boundary-layer flow considered here is still in a laminar regime although the transition to turbulence has started. This is because in a fully turbulent flow regime the phase trajectory of the temperature would not form a loop.

Figure 18(b) presents the streamwise profile of the standard deviation of the temperatures at the quasi-steady stage and (b) streamwise profiles of the local Nusselt numbers of the heated surface in the perturbed (solid line) and unperturbed (dashed line) conditions. DE denotes deviation region and OS denotes oscillation region.
the sinusoidal-like oscillations start to distort (refer to figure 17). Accordingly, the critical height determined from the appearance of the first peak of the standard deviation is recognized as the transition height $y_{tr}$ of the thermal boundary layer.

As it is widely believed that a transition to turbulence will result in enhancement of heat transfer, the profiles of the local Nusselt number on the heated surface under the perturbed and unperturbed conditions are plotted in figure 18(b) to further verify the above observed transition. Note that the local Nusselt number $Nu$ is estimated in the quasi-steady stage and calculated from

$$Nu = -\left.\frac{\partial \theta}{\partial x}\right|_{x=0}.$$  

(3.4)

It is seen in figure 18(b) that the profile of the local Nusselt number $Nu$ of the perturbed case ($f_{pt} = f_c$) can be broadly divided into two regions: an oscillation region (OS), where the local Nusselt number of the perturbed case approximately oscillates about the local Nusselt number of the unperturbed case; and a deviation region (DE), where the local Nusselt number of the perturbed case apparently deviates from that of the unperturbed case. The appreciable positive deviation beyond approximately $y = 0.58$ further confirms that the transition to nonlinear regime occurs around $y = 0.58$, which is consistent with the observation in figure 18(a). The overlap between the positive deviation region (DE) in figure 18(b) and the nonlinear regime shown in figure 18(a) implies that heat transfer enhancement can be achieved in the nonlinear regime by triggering resonance of a thermal boundary layer.

The effect of the perturbation frequency on the transition height $y_{tr}$ is further examined in figure 19. It is seen that the perturbation at the characteristic frequency of the boundary layer results in the lowest transition height, i.e. the earliest transition to turbulence. Perturbations at frequencies deviated away from the characteristic frequency result in later transition of the thermal boundary layer. The larger the deviation of the perturbation frequency from the characteristic frequency, the later the transition occurs. It is also anticipated that the transition will not occur if the
perturbation frequency is beyond a particular band of frequency, which will be briefly discussed below.

3.6. The effects of resonance on heat transfer

One of the major purposes of this investigation is to determine the heat transfer property of the thermal boundary layer subjected to various perturbations. First we examine the effect of the amplitude of the perturbation on heat transfer. For this purpose, we calculate the Nusselt numbers over the downstream half of the vertical surface, i.e. $0.5 \leq y \leq 1$, where the transition to turbulence is most likely to occur. We define an enhancement factor of the average Nusselt number $\varepsilon_{\text{Nu}}$ as

$$\varepsilon_{\text{Nu}} = \frac{\bar{Nu}_{\text{pt}} - \bar{Nu}_0}{\bar{Nu}_0},$$

where $\bar{Nu}_{\text{pt}}$ and $\bar{Nu}_0$ are the average Nusselt numbers of the heated surface in the perturbed and unperturbed cases, respectively. The average Nusselt number $\overline{Nu}$ is calculated from

$$\overline{Nu} = 2 \int_{0.5}^{1} \left. - \frac{\partial \theta}{\partial x} \right|_{x=0} \mathrm{d}y.$$  

Figure 20 illustrates the heat transfer enhancement factor as a function of the perturbation amplitude for the thermal boundary layer perturbed at the characteristic frequency ($f_{\text{pt}} = f_c$) for $Ra = 3.6 \times 10^{10}$ and an arbitrary frequency ($f_{\text{pt}} = 8 \times 10^4$) outside the resonance frequency band, respectively. The transition height for the thermal boundary layer perturbed at $f_{\text{pt}} = f_c$ with different perturbation amplitude is also plotted. It is seen in figure 20 that the transition height reduces and the enhancement factor increases with the increase of the perturbation amplitude when the
thermal boundary layer is perturbed at the characteristic frequency. The enhancement factors and their corresponding transition heights are linked by dashed lines for clarity. The enhancement factor approximately reaches 46% with a perturbation amplitude of $A_s$. The effect of the perturbation amplitude on the enhancement factor is very weak if the thermal boundary layer is perturbed at $f_{pt} = 8 \times 10^4$, and no apparent transition to turbulence is observed at this perturbation frequency. This distinct variation between the two perturbed cases is related to the resonance-induced advancement of laminar to turbulent transition, which occurs when the thermal boundary layer is perturbed at the characteristic frequency but does not occur when perturbed at a frequency outside the resonance frequency band.

It is worth noting that, if the perturbation amplitude is less than a certain value, such as the lowest two values shown in figure 20 (corresponding to perturbation amplitudes of $A_s/30$ and $A_s/60$, respectively), the above-described transition of the thermal boundary layer does not occur even if the boundary layer is perturbed at its characteristic frequency. The critical perturbation amplitude may depend on the Rayleigh number and the perturbation frequency. The determination of the critical perturbation amplitude is beyond the scope of this investigation.

Since the artificial perturbation introduced into the flow requires external energy input, we define a ratio of net heat transfer enhancement as

$$\varepsilon_{\text{net}} = \frac{E_{pt} - E_s - E_0}{E_0},$$

(3.7)

where $E_s$ is the energy consumed by the perturbation source over one forcing cycle ($\Delta t = 1/f_{pt}$) in the quasi-steady stage, $E_{pt}$ is the total heat dissipation over the same cycle from the downstream half of the surface when the boundary layer is perturbed at its characteristic frequency, and $E_0$ is the heat dissipation over the equivalent time ($\Delta t = 1/f_{pt}$) in the steady stage, from the same part of the surface but without perturbation. Here $E_s$ and $E_{pt}$ are evaluated by integration over a full sinusoidal cycle of the perturbation source (for single-mode perturbations only); $E_0$ is calculated over the same period of time for the unperturbed boundary layer. The results are shown for various Rayleigh numbers and two different perturbation amplitudes in figure 21.
All the flows at different Rayleigh numbers are perturbed by their corresponding characteristic frequencies that are determined from correlation (3.2).

It is seen that the ratio of net heat transfer enhancement with both perturbation amplitudes increases with $Ra$. This suggests that the resonance-based approach for enhancing heat transfer works better at larger Rayleigh numbers at which the thermal boundary layers are more unstable. For a given Rayleigh number, a smaller perturbation amplitude results in a lower ratio of net heat transfer enhancement. This is due to the delayed boundary-layer transition to turbulence, as discussed previously (refer to figure 20).

4. Conclusions

The instability and resonance characteristics of the thermal boundary layer adjacent to an isothermally heated surface are investigated by means of direct stability analyses. The main objective of this study is to investigate the resonance characteristics of the thermal boundary layer and its impact on heat transfer.

When a thermal boundary layer is randomly perturbed, three distinct frequency regions of the boundary layer have been identified, which include an upstream low-frequency region, a transitional region (with both high and low frequencies) and a downstream high-frequency region. The characteristic frequency of the thermal boundary layer is determined from the downstream high-frequency region. It is also found that the dependence of the characteristic frequency on $Ra$ can be described by $f_c = 0.0136Ra^{2/3}$.

The single-mode perturbation experiments described in § 3.2 have presented numerical evidence for the occurrence of resonance in the thermal boundary layer. The resonance can be triggered by single-mode perturbations at frequencies within the characteristic frequency band of the thermal boundary layer (i.e. the high-frequency band). This frequency band, which results in strong amplification of perturbations and is referred to as the resonance frequency band $B_r$, has been determined for $Ra$ over the range of $2.3 \times 10^8$ to $4.5 \times 10^9$.

The present study has also revealed that the transition height of the thermal boundary layer from the linear to the nonlinear response regime depends on both the perturbation frequency and amplitude. For a fixed perturbation amplitude above a critical value, the transition can be advanced mostly by perturbing the thermal boundary layer at its characteristic frequency. It is also found that heat transfer through the downstream half of the surface may be enhanced by up to 46% for $Ra = 3.6 \times 10^{10}$ when the thermal boundary layer is perturbed at the characteristic frequency. When the energy consumption by the perturbation source is taken into account, the net heat transfer enhancement can be still as high as 44%. This significant enhancement of heat transfer is a result of the resonance-induced advancement of laminar–turbulent transition.

This study is concerned with the resonance characteristics of the natural convection boundary-layer flows for the purpose of enhancing heat transfer. Whilst the study has demonstrated significant enhancement of heat transfer by perturbing the thermal boundary layer at its characteristic frequency, the enhancement is achieved by artificially introducing perturbations into the flow. The next step of this research will be focused on identifying a mechanism that can passively produce the desirable perturbations.
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REFERENCES


